Panofsky-Wenzel Theorem

Classical Mechanics and Electromagnetism for Accelerators

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Introduction

The Panofsky-Wenzel (PW) Theorem was originally used to describe the relationship between the longitudinal and transverse wake fields produced by a beam as it travelled through a device. A particle, as it travels through a device and encounters an aperture (or some other perturbation), generates a wave excitation that can produce an integrated longitudinal and transverse momentum kick. General relations of electromagnetic theory can be used to describe the relationship between the kicks, and this is what is stated in the PW theorem. This theorem can be generalized to include excitations, or fields and potentials, from external devices such as linacs, RF deflectors, etc. It is within this framework that we will work to derive the result.

Assumptions

We will be using the rigid beam approximation in the derivation. This states that an ultra relativistic particle is traveling along the z-axis

$$\boldsymbol{v} = v_0 \hat{z} \tag{1}$$

with a constant velocity in the paraxial limit. We also assume that we have a black box with external fields that are completely contained within its boundaries. Therefore, the potentials and fields from this external element, regardless of what it is, vanish outside the boundary region, R.

Field Effects

The fields interact with the particle through the Lorentz force equation:

$$\boldsymbol{F} = q(\boldsymbol{E} + \boldsymbol{v} \times \boldsymbol{B}) \tag{2}$$

The fields can be expressed in terms of the scalar and vector potential:

$$\begin{aligned} \boldsymbol{E} &= -\boldsymbol{\nabla}\phi - \frac{\partial \boldsymbol{A}}{\partial t} \\ \boldsymbol{B} &= \boldsymbol{\nabla} \times \boldsymbol{A}. \end{aligned} \tag{3}$$

For a particle traveling along the z-axis we can re-write the magnetic contribution

$$\boldsymbol{v} \times \boldsymbol{B} = v_0 \hat{z} \times (\boldsymbol{\nabla} \times \boldsymbol{A}) \tag{4}$$

using the following general relationship:

$$\boldsymbol{\nabla}(\boldsymbol{a} \cdot \boldsymbol{b}) = \boldsymbol{a} \times (\boldsymbol{\nabla} \times \boldsymbol{b}) + \boldsymbol{b} \times (\boldsymbol{\nabla} \times \boldsymbol{a}) + (\boldsymbol{a} \cdot \boldsymbol{\nabla})\boldsymbol{b} + (\boldsymbol{b} \cdot \boldsymbol{\nabla})\boldsymbol{a} \tag{5}$$

for arbitrary vectors $\boldsymbol{a}$ and $\boldsymbol{b}$. We therefore find:

$$v_0 \hat{z} \times (\boldsymbol{\nabla} \times \boldsymbol{A}) = \boldsymbol{\nabla}(\boldsymbol{A} \cdot v_0 \hat{z}) - \boldsymbol{A} \times (\boldsymbol{\nabla} \times v_0 \hat{z}) - (\boldsymbol{A} \cdot \boldsymbol{\nabla}) v_0 \hat{z} - (v_0 \hat{z} \cdot \boldsymbol{\nabla})\boldsymbol{A}. \tag{6}$$

The second and third term vanish and we are left with

$$v_0 \hat{z} \times (\boldsymbol{\nabla} \times \boldsymbol{A}) = v_0 \boldsymbol{\nabla} A_z - v_0 \frac{\partial \boldsymbol{A}}{\partial z}. \tag{7}$$

If we use the full convective derivative

$$\frac{\mathrm{d}\boldsymbol{A}}{\mathrm{d}t} = \frac{\partial \boldsymbol{A}}{\partial t} + (\boldsymbol{v} \cdot \boldsymbol{\nabla})\boldsymbol{A} = \frac{\partial \boldsymbol{A}}{\partial t} + v_0 \frac{\partial \boldsymbol{A}}{\partial z} \tag{8}$$

and change variables

$$v_0 \frac{\mathrm{d}\boldsymbol{A}}{\mathrm{d}z} = \frac{\mathrm{d}\boldsymbol{A}}{\mathrm{d}t} \tag{9}$$

we find

$$\boldsymbol{F} = q\left(-\boldsymbol{\nabla}\phi + v_0 \boldsymbol{\nabla} A_z - v_0 \frac{\mathrm{d}\boldsymbol{A}}{\mathrm{d}z}\right). \tag{10}$$

The full momentum transfer through a region R can be found by integrating this equation:

$$\Delta \boldsymbol{p} = q \int_R \left[-\frac{\mathrm{d}\boldsymbol{A}}{\mathrm{d}z} + \boldsymbol{\nabla}\left(A_z - \frac{\phi}{v_0}\right)\right] \mathrm{d}z. \tag{11}$$

When evaluating the endpoints of the integration, $\boldsymbol{A} = 0$ for an isolated system, where $\boldsymbol{E}$ and $\boldsymbol{B}$ vanish on the boundary. We are therefore left with

$$\Delta \boldsymbol{p} = \boldsymbol{\nabla}\left[q \int_R \left(A_z - \frac{\phi}{v_0}\right) \mathrm{d}z\right] \equiv \boldsymbol{\nabla}\Omega. \tag{12}$$

Since $\boldsymbol{\nabla} \times (\boldsymbol{\nabla}\Omega) = 0$ we arrive at

$$\boldsymbol{\nabla} \times (\Delta \boldsymbol{p}) = 0. \tag{13}$$

This is the Panofsky-Wenzel Theorem. This can be re-written in as

$$\boldsymbol{\nabla}_\perp (\Delta p_z) = \frac{\partial (\Delta \boldsymbol{p}_\perp)}{\partial z}. \tag{14}$$

This is a statement that, for a deflection kick to occur there must be transverse variation of the longitudinal kcik imparted by the fields of the isolated system under consideration.